An Introduction to Acceleration in Special Relativity

www.zitterbug.net/future/future815.html

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The following is a derivation of a relationship between the time that passes with a rest frame (*Earth*)[1] and the time that passes within a moving frame (*a starship*), that starting from rest, moves with a predetermined constant acceleration[2] to a predetermined destination and back.

We begin with the Lorentz Transformations[3] for a point P in S and moving frame S', where S' moves with velocity v in the positive x direction. The Lorentz Transformations from the Earth frame S to ship frame S' is stated first, followed by the Inverse Lorentz Transformations ($S' \rightarrow S$). The differentials of these transformations are found and rearranged to find the time transformation equations between frames where the point accelerates from rest at a constant rate.

The **Lorentz Transformations** for point $P(x', y', z', t')$ where S' moves with speed v in the positive x direction, t is the rest frame time, c is the speed of light, and $\gamma^{-1} = \sqrt{1 - v^2/c^2}$,

$$x' = \frac{x - vt}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(x - vt\right) \tag{1a}$$

$$y' = y \tag{1b}$$

$$z' = z \tag{1c}$$

$$t' = \frac{t - xv/c^2}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(t - xv/c^2\right) \tag{1d}$$

The **Inverse Lorentz Transformations** for point $P(x, y, z, t)$,

[1] Though Earth is obviously not an inertial reference frame, the effects of the Earth's rotation, orbital motion etc., are omitted and assumed negligible in our us of *Earth* as an inertial rest frame.

[2] L. Marder, *Time and the Space-Traveller* : "Imagine the space-ship to run its motors at a fixed setting of the controls, so that propellant is ejected at a constant rate and constant velocity relative to the ship. If, during the period of acceleration the mass of propellant used is negligible, compared with the total mass of the ship, its occupants and the remaining fuel, etc., then the motion of the ship should satisfy any reasonable criterion of uniform acceleration (p.93)." In reality, using conventional rockets to accelerate an object, even the size of a model rocket, for more than a few seconds would require an astronomically large amount of fuel (this problem and getting around it is a subject for later review).

[3] There are several very good introductions and derivations of the Lorentz Transformations (see References).

$$x = \frac{x' + vt'}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(x' + vt'\right) \tag{2a}$$

$$y = y' \tag{2b}$$

$$z = z' \tag{2c}$$

$$t = \frac{t' + x'v/c^2}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(t' + x'v/c^2\right) \tag{2d}$$

Taking the differentials of (2a) and (2d),

$$dx = \frac{dx' + vdt'}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(dx' + vdt'\right) = \gamma\left(v'_x + v\right)dt' \tag{3a}$$

$$dt = \frac{dt' + \left(v/c^2\right)dx'}{\sqrt{\left(1 - v^2/c^2\right)}} = \gamma\left(1 + v'_x v/c^2\right)dt' \tag{3b}$$

where $v'_x = dx'/dt'$ is the speed of P in S'.

Dividing (3a) by (3b) to find the speed v_x of the point in S,

$$\frac{dx}{dt} = \frac{\gamma\left(v'_x + v\right)dt'}{\gamma\left(1 + v'_x v/c^2\right)dt'} = \frac{v'_x + v}{1 + v'_x v/c^2},$$

$$v_x = \frac{v'_x + v}{1 + v'_x v/c^2} \tag{4}$$

Differentiating v_x to find the acceleration a_x of the point in S,

$$a_x \equiv \frac{d}{dt}v_x = \frac{d}{dt}\left(\frac{v'_x + v}{1 + v'_x v/c^2}\right) = \frac{dv'_x/dt}{1 + v'_x v/c^2} - \frac{\left(v'_x + v\right)\left(v/c^2\right)}{\left(1 + v'_x v/c^2\right)^2}\left(\frac{dv'_x}{dt}\right) = \frac{dv'_x/dt}{\gamma^2\left(1 + v'_x v/c^2\right)^2},$$

Substituting $dt = \gamma\left(1+v_x' v/c^2\right)dt'$ in the above,

$$a_x = \frac{dv_x'/dt}{\gamma^2\left(1+v_x'v/c^2\right)^2} = \frac{dv_x'/dt'}{\gamma^3\left(1+v_x'v/c^2\right)^3} = \frac{a_x'}{\gamma^3\left(1+v_x'v/c^2\right)^3},$$

$$a_x = \frac{a_x'}{\gamma^3\left(1+v_x'v/c^2\right)^3} \tag{5a}$$

The acceleration of point P in S', $a_x' = dv'/dt'$ is held constant; to emphasize this and to simply the writing, the acceleration of the point in the moving frame S' will be represented with "*a*." The ship and its contents should thought of as the point P accelerating in S', while P and S' momentarily have the same *instantaneous* speed *v*. Summarizing[4],

$a_x' \equiv a$, $v_x' = 0$, and $v_x = v$, from (4).

Therefore (5a) can be written,

$$\begin{aligned} a_x &= \frac{a}{\gamma^3}, \\ \frac{dv_x}{dt} &= \left(1-v^2/c^2\right)^{3/2} a \end{aligned} \tag{5b}$$

Integrating (5b),

$$\begin{aligned} \int_0^v \frac{dv}{\left(\sqrt{1-v^2/c^2}\right)^3} &= a\int_0^t dt, \\ \frac{v}{\sqrt{1-v^2/c^2}} &= at \end{aligned} \tag{5c}$$

Rearranging (5c),

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[4] It might help to recall that a point moving in simple harmonic motion can have a non-zero instantaneous acceleration and instantaneous speed of zero.

$$\frac{v^2}{1-v^2/c^2}=a^2t^2\rightarrow v^2\left(1+a^2t^2/c^2\right)=a^2t^2\rightarrow v=\frac{at}{\sqrt{1+a^2t^2/c^2}},$$

$$v=\frac{at}{\sqrt{1+a^2t^2/c^2}}, \tag{5d}$$ [5]

Rearranging again to find another useful form of (5c),

$$at=\frac{v}{\sqrt{1-v^2/c^2}}\rightarrow\left(\frac{1}{at}\right)(v)=\sqrt{1-v^2/c^2}\rightarrow\left(\frac{1}{at}\right)\left(\frac{at}{\sqrt{1+a^2t^2/c^2}}\right)=\sqrt{1-v^2/c^2},$$

$$\sqrt{1-v^2/c^2}=\frac{1}{\sqrt{1+a^2t^2/c^2}} \tag{5e}$$

Integrating (5e) to find the time transform equations with constant acceleration,

$$\tau=\int_0^t dt'=\int_0^T\left(1-v^2/c^2\right)^{\frac{1}{2}}dt=\int_0^t\frac{dt}{\sqrt{1+a^2t^2/c^2}}=\frac{c}{a}\ln\left(\frac{a}{c}t+\sqrt{1+\frac{a^2t^2}{c^2}}\right)=\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}t\right),$$

Time Transform with Constant Acceleration

$$\tau=\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}t\right) \tag{1α}$$

Finding the inverse of (1α),

$$t=\frac{c\left(e^{a\tau/c}-e^{-aT'/c}\right)}{2a}=\frac{c}{a}\sinh\left(\frac{a}{c}\tau\right),$$

Inverse Time Transform with Constant Acceleration

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[5] Note that in terms of the ship's time (a.k.a. *proper time*, τ), equation (5d) becomes
$v=at\big/\sqrt{1+a^2t^2/c^2}=a\left[(c/a)\sinh(a\tau/c)\right]\big/\cosh(a\tau/c)=c\tanh(a\tau/c)$.

$$t = \frac{c}{a}\sinh\left(\frac{a}{c}\tau\right) \tag{1β}$$

where t is the time within S (the rest frame), a is the acceleration of P in the moving frame, c is the speed of light, and τ is time within the moving frame S'.

Journey to the Stars

NASA's newly discovered Earth-like planet Kepler-452b is about 1,400 light-years.[6] Though it's over a thousand years away, equation 1α can be used to determine the time it would take to reach Kepler-452b travelling at a constant velocity of 1g in both the frame of the Earth (obviously over 1000 years) and in the frame of starship (amazingly, within an average life time). The journey is divided up into four parts, accelerating to the midpoint of the outward destination, turning the engines around to decelerate to rest at the destination, accelerating back to the midpoint, turning the engines again at the midpoint to return to rest on an Earth's that over 2000 years older that when the ship left it.

To find the rest frame time (time on Earth) it would take to reach a given distance, in a ship travelling with constant acceleration, equation (5d) is integrated to find the time needed to reach the ship's turn-around (midway to the destination). This time is then doubled to find the total rest frame time to reach its destination. Since the engine turn-around time is assumed negligible and ship decelerates at the same rate as it accelerates to the midpoint, by symmetry the time of the ship's constant deceleration to rest is the same as the outbound time.

Rearranging (5d), integrating to find the distance traveled *x* with constant acceleration, staring from rest after time *t*,

$$v = \frac{dx}{dt} = \frac{at}{\sqrt{1+a^2t^2/c^2}} \to dx = \frac{atdt}{\sqrt{1+a^2t^2/c^2}} \to \int_0^x dx = \int_0^t \frac{atdt}{\sqrt{1+a^2t^2/c^2}},$$

$$x = \left(c^2/a\right)\left(\sqrt{1+a^2t^2/c^2} - 1\right) \tag{6a}$$

[6] NASA – Kepler-452b-Earth-Like-Planet

Let X represent the outward bound distance, then multiplying (6a) by 2 and solving for time *t*,

$$\mathrm{X} = 2\left(c^2/a\right)\left(\sqrt{1+a^2t^2/c^2}-1\right) \tag{6b}$$

$$x = \left(c^2/a\right)\left(\sqrt{1+a^2t^2/c^2}-1\right) \to \frac{ax}{c^2}+1 = \sqrt{1+a^2t^2/c^2} \to t^2 = \frac{x^2}{c^2}+2\frac{x}{a},$$

$$t = \sqrt{\frac{x^2}{c^2}+\frac{2x}{a}} \tag{7a}$$

If the destination distance X (the distance to the midpoint engine turnaround is $\mathrm{X}/2$), then total time T_{X} to the destination is twice the time t_{X} to the midpoint (twice t_{X}):

$$T_{\mathrm{X}} = 2t_{\mathrm{X}} = 2\sqrt{\frac{1}{c^2}\left(\frac{\mathrm{X}}{2}\right)^2+\frac{2}{a}\left(\frac{\mathrm{X}}{2}\right)} = 2\sqrt{\frac{\mathrm{X}^2}{4c^2}+\frac{\mathrm{X}}{2}},$$

$$T_{\mathrm{X}} = 2\sqrt{\frac{\mathrm{X}^2}{4c^2}+\frac{\mathrm{X}}{2}} \tag{7b}$$

$$t = \sqrt{\frac{x^2}{c^2}+\frac{2x}{a}}$$

$$t' = \left(c/a\right)\sinh^{-1}\left[\left(a/c\right)t\right]$$

$$t' = \frac{c}{a}\sinh^{-1}\left[\frac{a}{c}\sqrt{\frac{x^2}{c^2}+\frac{2x}{a}}\right]$$

Substituting half of 1α into 7a and multiplying by 2,

$$T' = 2\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{X^2}{4c^2}+\frac{X}{2}}\right)$$

If 1α and 1β are used to find the rest frame round-trip T and moving frame round-trip time T', respectively, then the total of each time has to be understood as the sum of *four equal blocks of time* – by symmetry, two equal blocks of outward trip time and two equal blocks of return time,

$$T' = 4\cdot\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}\cdot\frac{T}{4}\right) \tag{8a}$$

and

$$T = 4\cdot\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}\cdot\frac{T'}{4}\right). \tag{8b}$$

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Schlegel, Richard (1968) *Time and the Physical World*. Dover Publications, Inc. (1961, Michigan State University Press): New York.

LINKS OF INTEREST

Susskind Lectures on Relativistic Kinematics

George Smoot on Relativity: Physics 139 Relativity, UC Berkeley

The Original Usenet Physics FAQ: The Relativistic Rocket

scienceworld.wolfram: "Proper Time"

scienceworld.wolfram: "Velocity Four-Vector"

Leonard Susskind on Special Relativity (YouTube)

Restarting the LHC: Why 13 Tev?

NASA – Kepler-452b-Earth-Like-Planet

Summary of equations used in www.zitterbug.net

STARSHIP TIME TO EARTH TIME:	$T = 4 \cdot \frac{c}{a} \sinh^{-1}\left(\frac{a}{c} \cdot \frac{T'}{4}\right)$	(8a)
EARTH TIME TO STARSHIP TIME:	$T' = 4 \cdot \frac{c}{a} \sinh^{-1}\left(\frac{a}{c} \cdot \frac{T}{4}\right)$	(8b)
EARTH TIME MINUS STARSHIP TIME:	$T - T' = T - 4\frac{c}{a}\sinh^{-1}\left(\frac{a}{4c}T\right) = 4\frac{c}{a}\sinh^{-1}\left(\frac{a}{4c}T'\right) - T'$ The Newton-Raphson method is used to solve for T' given T or T given T'.	
MAXIMUM SPEED:	$v = \frac{at}{\sqrt{1 + a^2t^2/c^2}}$	(5d)
FARTHEST DISTANCE:	$\mathrm{X} = 2\left(c^2/a\right)\left(\sqrt{1 + a^2t^2/c^2} - 1\right)$	
EARTH TIME TO DESTINATION:	$T_{\mathrm{X}} = 2\sqrt{\frac{\mathrm{X}^2}{4c^2} + \frac{\mathrm{X}}{2}}$	(7b)
SHIP TIME TO DESTINATION:	$T'_{\mathrm{X}} = 2\frac{c}{a}\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{\mathrm{X}^2}{4c^2} + \frac{\mathrm{X}}{2}}\right)$	(6b)

Summary of useful forms of (5c)

$$at = \frac{v}{\sqrt{1 - v^2/c^2}} \tag{5c}$$

$$v = \frac{at}{\sqrt{1 + a^2t^2/c^2}} \tag{5d}$$

$$\gamma = \left(\sqrt{1 - v^2/c^2}\right)^{-1} = \sqrt{1 + a^2t^2/c^2} \tag{5e}$$

$$dt' = \frac{dt}{\sqrt{1 + a^2t^2/c^2}}$$

$$dt = dt' \cosh\left(at'/c\right)$$